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Collineation Groups Preserving a Unital of a Projective Plane of Odd Order*

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INTRODUCTION

A unital embedded in a finite projective plane Π of order m^2 is a substructure of Π which forms a $2-(m^3+1, m+1, 1)$ design.

Unitals first originate in a finite projective plane Π, endowed with a unitary polarity ϑ. In such a situation the substructure of Π consisting of absolute points and non-absolute lines of ϑ is just a unital embedded in Π, as shown by Seib in [20]. Nevertheless there are examples of unitals which are not related to any polarity in the plane [4].

The "classical" unital consists of absolute points and non-absolute lines of a hermitian polarity of a desarguesian plane of order m^2. It is left invariant by a collineation group isomorphic to $P\Gamma U(3, m^2)$, which acts 2-transitively on its points. Conversely, Hoffer [13] showed that a unital of a projective plane of order m^2, which is invariant under a collineation group isomorphic to $PSU(3, m^2)$, is classical.

Subsequently, Kantor [16] proved that if a unital $\mathcal{U}$ is invariant under a collineation group G acting transitively on the flags consisting of a point

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and a secant line of $\mathscr{U}$, then, again, $\mathscr{U}$ is classical and $PSU(3, m^2) \leqslant G \leqslant P\Gamma U(3, m^2)$. Actually, it suffices to suppose that G is transitive on the secants of $\mathscr{U}$, since that implies the flag transitivity, as noticed by Camina and Gagen in [6].

It is also well known [7] that the flag transitivity of G on $\mathscr{U}$ implies the primitivity of G on the points of $\mathscr{U}$, the converse being not true, since a classical unital in the desarguesian plane of order 9 is invariant under a collineation group isomorphic to $PGL(2, 7)$, which is primitive on its points, but not flag transitive.

The aim of this paper is to carry on the study of collineation groups which preserve unitals embedded in projective planes, limiting ourselves to the odd order case.

In Section 2 we obtain several results about the structure of 2-groups. The main result is the following.

THEOREM A. *Let G be a 2-group of collineations of a finite projective plane Π of odd order m^2 which preserves a unital of Π. If $m \equiv 1(4)$, then G has 2-rank at most 3.*

Combining results of Section 2 with the classification of simples groups of 2-rank at most 3 (see [9]), it is possible to obtain rather strong results about the structure of non-abelian simple groups which preserve an embedded unital. Precisely, we have

THEOREM B. *Let G be a non-abelian simple group of collineations of a finite projective plane Π of odd order m^2 which preserves a unital of Π. Then one of the following holds:*

(i) *$m \equiv 1(4)$ and either $G \simeq PSL(2, q)$ with q odd or $G \simeq PSU(3, q^2)$ with $q \equiv 1(4)$ or $G \simeq A_7$ or $PSL(2, 8)$.*

(ii) *$m \equiv 3(4)$ and either G is a group of homologies with the same centre and the same axis or $G \simeq PSL(2, q)$, $PSU(3, q^2)$, $Sz(q)$, or A_7.*

Moreover, involutions in G are homologies except possibly when $m \equiv 1(4)$ and $G \simeq PSL(2, q)$, A_7, or $PSL(2, 8)$.

These results allow us to make progress on the study of collineation groups which act transitively on the points of an embedded unital. In Section 4 we prove the following result.

THEOREM C. *Let G be a collineation group of a finite projective plane Π of odd order which preserves a unital $\mathscr{U}$ of Π. If G is transitive on the points of $\mathscr{U}$ and the socle S of G has even order then one of the following holds:*

(j) $S \simeq PSU(3, q^2)$, *q odd, Π is desarguesian of order q^2, and $\mathscr{U}$ is classical.*

(jj) $S \simeq PSL(2, 7)$, *Π is desarguesian of order* 9, *and $\mathscr{U}$ is classical.*

(jjj) $S \simeq A_7$, *S acts transitively on the points of $\mathscr{U}$, and Π has order* 25.

Furthermore. $S \leqslant G \leqslant \mathrm{Aut}(S)$.

The proof makes use of Theorems A and B and of Hering's theory on strongly irreducible groups of collineations [10]. A restricted number of admissible situations is settled by a direct investigation. We point out that the case $S \simeq A_7$ actually occurs in the desarguesian plane of order 25.

As an easy corollary of Theorem C, we obtain:

THEOREM D. *Let G be a collineation group of a finite projective plane Π of odd order m^2 which acts primitively on the points of a G-invariant unital $\mathscr{U}$ of Π. Then Π is desarguesian and $\mathscr{U}$ is a classical unital. Moreover either* $PSU(3, m^2) \leqslant G \leqslant P\Gamma U(3, m^2)$ *or* $m = 3$ *and* $G \simeq PGL(2, 7)$.

We notice that a similar investigation for projective planes of even order still has not been developed and the characterization problem under the primitivity assumption is still open.

1. PRELIMINARY RESULTS

We shall use standard notation. The necessary background about projective planes may be found in [7]. Let Π be a projective plane. If H is a collineation group of Π, then $F(H)$ denotes the substructure of Π consisting of those points and lines which are fixed by each element of H. We shall write $F(\alpha)$ instead of $F(\langle\alpha\rangle)$ for any collineation α. *A triangular group K of homologies* of Π is an elementary abelian 2-group of order 4 (Kein group) of collineations of Π whose involutions are homologies and such that $F(K)$ is a triangle.

LEMMA 1.1. *The following hold*:

(1) *If K is a Klein group of collineations of Π whose involutions are homologies, then either all homologies in K have the same centre and the same axis, or K is a triangular group of homologies.*

(2) *If K is a triangular group of homologies and (X, x) is a point-line pair consisting of a vertex of $F(K)$ and its opposite side, then Π possesses exactly one (X, x)-involutorial homology.*

(3) *There is no abelian collineation group of order* 8 *generated by three involutorial homologies not all having the same centre and the same axis.*

For a proof see [16, Lemmas 2.1 and 2.2].

We also recall that, following Hering [10], a collineaton group G of Π is called *strongly irreducible* if G does not fix any point, line, triangle, or proper subplane of Π.

For what concerns finite groups the reader is referred to [8, 14]. For simple groups see also [9]. Here we only recall that a group G is said to be of 2-rank h if h is the maximum rank of an abelian 2-subgroup of G.

Simple groups of 2-rank $\leqslant 3$ are well known.

Result 1.2. Let G be a finite, non-abelian, simple group of 2-rank at most 3. Then either

(i) $G \simeq G_2(q)$, ${}^3D_4(q)$, ${}^2G_2(3^n)$, $PSL(2, 8)$, $PSU(3, 8^2)$, $Sz(8)$, M_{12}, J_1, ON, where q and n are odd, $n > 1$, and G has 2-rank 3, or

(ii) $G \simeq PSL(2, q)$ $(q \geqslant 5)$, $PSL(3, q)$, $PSU(3, q^2)$, $PSU(3, 4^2)$, A_7, M_{11}, where q is odd, and G has 2-rank 2.

Moreover, G has only one class of involutions except when $G \simeq M_{12}$, in which case G has two classes of involutions.

Proof. See [9, Theorems 1.86 and 2.168] for assertions (i) and (ii). The number of involution classes is well known (e.g., see [9] for $G_2(q)$ and ${}^3D_4(q)$, [15] for ${}^2G_2(3^n)$ and $Sz(8)$, [19] for sporadic simple groups).

Now, we enumerate several esults about ovals of Π, which will be useful in the following. Assume Π has odd order n. An oval Ω of Π is a set of $n+1$ points of Π, no three of which are collinear. A line l of Π is called an external line, a tangent, or a secant of Ω according to whether $|l \cap \Omega| = 0$, 1, or 2. Each point of Π, not in Ω, lies in either two or no tangents of Ω and it is called an external point or an internal point, respectively. For more details see [7].

Let G be a collineation group of Π which leaves Ω invariant.

Result 1.3. If σ is an involution of G, then one of the following holds:

(1) σ induces an odd permutation on Ω. Moreover,

(a) if σ is a (L, l)-homology, then L is an internal point and l is an external line of Ω, or L is an external point and l is a secant of Ω, according to whether $n \equiv 1(4)$ or $n \equiv 3(4)$;

(b) if σ is a Baer collineation, then either $\Omega \cap F(\sigma) = \varnothing$, or $\Omega \cap F(\sigma)$ is an oval of $F(\sigma)$ and $\sqrt{n} \equiv 3(4)$.

(2) σ induces an even permutation on Ω. Moreover,

(a) if σ is a (L, l)-homology, then L is an external point and l is a secant of Ω, or L is an internal point and l is an external line of Ω, according to whether $n \equiv 1(4)$ or $n \equiv 3(4)$;

(b) if σ is a Baer collineation, then $\Omega \cap F(\sigma)$ is an oval of $F(\sigma)$.

In addition, when σ is a Baer collineation, the following hold:

(i) if $\Omega \cap F(\sigma) = \emptyset$, then $(n+1)/2$ lines of $F(\sigma)$ are secants of Ω, while the remaining $(\sqrt{n}+1)^2/2$ lines are external to Ω,

(ii) if $\Omega \cap F(\sigma)$ is an oval of $F(\sigma)$, then no line of $F(\sigma)$ is external to Ω.

For a proof see [2, Proposition 2.1 and 2.2].

Result 1.4. The following hold:

(1) Two distinct involutorial homologies in G have both distinct centres and distinct axes.

(2) If K is a Klein subgroup of G, then K contains an involutorial homology which induces an even permutation on Ω.

(3) An elementary abelian 2-subgroup E of G has order at most 4 in the following cases:

(i) E fixes two distinct point of Ω,

(ii) E fixes an external line of Ω.

Proof. For (1) and (2) see [2, Propositions 2.1 and 2.3]. We shall prove (3).

The assertion is trivial when $n \equiv 3(4)$ by (1) and Lemma 1.1(3). If $n \equiv 1(4)$, first assume E fixes two points A and B of Ω. By Result 1.3 and (1), E can contain at most one homology, namely that with axis AB. The assertion then follows from (2).

Now, assume E fixes an external line of Ω and let E_0 be the subgroup of E whose elements induce even permutations on Ω. By Result 1.3(2(b)) and (ii), the non-identical elements of E_0 are homologies. Moreover, by (1) and Lemma 1.1(3), $|E_0| \leqslant 4$. If $|E_0| = 4$, then the only lines fixed by E_0 are the three axes of homologies in E_0, none of which is external to Ω, being $n \equiv 1(4)$. Therefore, $|E_0| \leqslant 2$ and hence $|E| \leqslant 4$.

2. Properties of 2-Groups Acting on Unitals

Let Π be a finite projective plane of odd square order $n = m^2$. A *unital* $\mathcal{U}$ of Π consists of a set $\mathbf{U}$ of m^3+1 points and a set $\mathbf{L}$ of $m^2(m^2-m+1)$ lines of Π which make a 2-$(m^3+1, m+1, 1)$ design with respect to the same incidence relation of Π. A line l of Π is called a *tangent* or a *secant* of $\mathscr{U}$ according to whether l contains 1 or $m+1$ points of $\mathscr{U}$. It is very easy to see that when $P \in \mathrm{U}$, there are m^2 secants and one tangent of $\mathscr{U}$ through P, while when $P \notin \mathrm{U}$, there are m^2-m secants and $m+1$ tangents of $\mathscr{U}$ through P. Notice that no line of Π is disjoint from $\mathscr{U}$.

Let G be a collineation group of Π which leaves $\mathscr{U}$ invariant.

LEMMA 2.1. *If σ is a Baer involution of G, then the following hold*:

(1) $\Omega_\sigma = F(\sigma) \cap \mathrm{U}$ *is an oval of* $F(\sigma)$,

(2) *the tangents of $\mathscr{U}$ lying in $F(\sigma)$ are exactly the tangents of Ω_σ*,

(3) *σ induces an odd permutation on the secants of $\mathscr{U}$ when $m \equiv 3(4)$*,

(4) *$\langle\sigma\rangle$ is the pointwise stabilizer of $F(\sigma)$ in G.*

For a proof of (1) and (2) see [20]. (3) and (4) are Lemmas 3.2 and 3.3 of [16].

LEMMA 2.2. *If σ is an involutorial (L, l)-homology of G, then the following hold*:

(1) *l is a secant of $\mathscr{U}$ and L is the common point of the $m+1$ tangents of $\mathscr{U}$, one through each point of $l \cap \mathrm{U}$.*

(2) *if τ is an involutorial (A, a)-homology such that $\sigma\tau = \tau\sigma$ and $a \neq l$, then the point $l \cap a$ does not lie in $\mathscr{U}$.*

If $m \equiv 1(4)$, then two distinct commuting involutorial homologies in G have both distinct centres and distinct axes.

Proof. (1) may be easily proved by counting arguments. (2) follows directly from (1), if one considers that $\sigma\tau$ is an homology with centre $l \cap a$.

Let σ_1, σ_2 be two distinct commuting involutorial homologies of G. Then either σ_1 and σ_2 have both distinct centres and distinct axes, or σ_1 and σ_2 have the same centre and the same axis. In the latter case $|\langle\sigma_1, \sigma_2\rangle| \mid m+1$ because of (1). This is a contradiction, being $m \equiv 1(4)$.

Now, we can prove Theorem A.

Proof of Theorem A. Let H be an elementary abelian subgroup of G. If H does not contain Baer collineations, then $|H| \leqslant 4$ by Lemmas 1.1 and 2.2. Assume H contains Baer collineations.

If H fixes two points of $\mathscr{U}$ and σ is a Baer collineation in H, then $H/\langle\sigma\rangle$ is isomorphic to a collineation group of $F(\sigma)$ fixing two points of $\Omega_\sigma = F(\sigma) \cap \mathrm{U}$. The result then follows from Result 1.4(3(i)).

Otherwise, H has an orbit $\{A, B\}$ of length 2 on U, since $m^3+1 \equiv 2(4)$. Put $l = AB$ and first suppose that H possesses an orbit of length 2 on $l \cap \mathrm{U}$, other than $\{A, B\}$. Then H possesses a subgroup H_0 of index at most 4, which fixes four points on $l \cap \mathrm{U}$. By Lemma 2.1 all non-trivial elements of H_0 are homologies with axis l. Therefore, $|H_0| \leqslant 2$ by Lemma 2.2 and hence $|H| \leqslant 2^3$.

Now, assume $\{A, B\}$ is the only orbit of length 2 on $l \cap \mathrm{U}$ and suppose that $|H| > 2^3$. Since the homologies of H generate a group of order at most 4, H must contain a Baer collineation σ which interchanges A and B. Clearly, l must be an external line of Ω_σ since otherwise $l \cap \Omega_\sigma$ would be an orbit of length 2 of H, different from $\{A, B\}$. By Result 1.4(3(ii)), the collineation group induced by H on $F(\sigma)$ has order at most 4, contrary to our assumption $|H| > 2^3$.

We may obtain some further restrictions.

PROPOSITION 2.3. *Let* $m \equiv 1(4)$. *The following hold*:

(1) *if G is a quaternion group, then the unique involution in G is a homology*,

(2) *if G is a dihedral* 2-*group and* $|G| \geqslant 16$, *then the central involution in G is a homology*.

Proof. (1) Let σ be the unique involution of G and assume σ is a Baer collineation.

The collineation group $\bar{G}$, induced by G on $F(\sigma)$, is a Klein group. Let $\bar{\tau} \in \bar{G}$, $\bar{\tau} \neq 1$. If $\bar{\tau}$ is not a Baer collineation of $F(\sigma)$ with fixed points on Ω_σ, then, by Result 1.3, $\bar{\tau}$ fixes a line l of $F(\sigma)$, which is external to Ω_σ. By Lemma 2.1(2), l is a secant of $\mathscr{U}$. Moreover, τ fixes $l \cap \mathrm{U}$ and has an orbit of length 2 on $l \cap \mathrm{U}$, because of $m+1 \equiv 2(4)$. It follows that τ^2 fixes two points of $l \cap \mathrm{U}$. This is absurd for l is external to Ω_σ.

Therefore $\bar{G}$ is a Klein group of Baer collineations; but this contradicts Result 1.4(2), and hence the assertion holds.

(2) Let $|G| = 2d$ and suppose that the central involution σ of D is a Baer collineation. By Lemma 2.1(4), the collineation group $\bar{G}$, induced by G on $F(\sigma)$, is a dihedral group of order d.

Suppose that $8 \mid d$. The central involution $\bar{\tau}$ of $\bar{G}$ is a Baer collineation of $F(\sigma)$ with fixed points on Ω_σ, as we have seen in the proof of (1). Nevertheless, $\bar{\tau}$ can be expressed as the product of two conjugate involutions of $\bar{G}$ which, obviously, must both be Baer collineations. This contradicts Result 1.4(2).

3. SIMPLE GROUPS PRESERVING AN EMBEDDED UNITAL

As in the previous section, let G be a collineation group of Π which leaves invariant a unital $\mathscr{U}$ of Π.

Suppose that G is non-abelian simple. When Π has order m^2 with $m \equiv 1(4)$, one can prove that the structure of G is very restricted though no assumption is made about the action of G on $\mathcal{U}$. When $m \equiv 3(4)$, one can obtain rather strong results only if we assume that G is not a group of homologies with the same centre and the same axis.

First of all, it seems likely that G must contain involutorial homologies. Here we prove the following theorem.

THEOREM 3.1. *If G is non-abelian simple then either G contains involutorial homologies or Π has order m^2 with $m \equiv 1(4)$ and $G \simeq A_7$, $PSL(2, 8)$, or $PSL(2, q)$ for q odd.*

Proof. Assume that all involutions in G are Baer collineations.

The case $m \equiv 3(4)$ cannot occur by Lemma 2.1(3). So, $m \equiv 1(4)$ and G has 2-rank $\leqslant 3$ by Theorem A. Therefore G is among the groups listed in Result 1.2. The proof develops into three steps.

(1) $G \not\simeq G_2(q)$, ${}^3D_4(q)$, M_{12}, $PSL(3, q)$, $PSU(3, q^2)$, M_{11}, $PSU(3, 8^2)$, $PSU(3, 4^2)$, ON.

Any such group contains a quaternion group (see [9, Theorem 2.4 and Theorem 1.63; 14, p. 245; 19]). By Proposition 2.3(1), this is impossible.

(2) $G \not\simeq {}^2G_2(3^n)$, J_1.

Suppose that G is isomorphic to any of the above groups and let T be a Sylow 2-subgroup of G. The following properties are well known (see [15, XI, Theorem 13.2; 19, Lemma 1.6]):

(a) T is elementary abelian of order 8,

(b) the involutions in T are conjugate under an element $\alpha \in N_G(T)$ of order 7,

(c) if $\sigma \in T$, $\sigma \neq 1$, then $C_G(\sigma) \simeq PSL(2, q) \times \langle \sigma \rangle$, where $q = 3^n$ for $G \simeq {}^2G_2(3^n)$ and $q = 5$ for $G \simeq J_1$.

We obtain a contradiction in two steps.

T contains both Baer involutions with fixed points on l and Baer involutions without fixed points on l for any T-invariant secant l of $\mathcal{U}$.

Suppose that T fixes two points A, B on l. Let $\sigma \in T$, $\sigma \neq 1$. We have that $T = T_0 \times \langle \sigma \rangle$ with $T_0 < PSL(2, q)$. Since all involutions in T_0 are conjugate under $PSL(2, q)$, T_0 acts on $F(\sigma)$ as a triangular group of homologies by Result 1.4(2). So, the only points of $F(\sigma)$ which are fixed by T_0 are the three homology centres, none of which lies in Ω_σ by Result 1.3. This is in contrast with our assumption that T fixes $A, B \in \Omega_\sigma$.

Therefore, we may assume that T does not fix any point on l. Since $m+1 \equiv 2(4)$, T possesses an orbit $\{A, B\}$ of length 2 on l. Let $\sigma \in T$ such that $A^\sigma = B$ and let T_1 be the subgroup of order 4 of T fixing both A and B. If there would be $C, D \in l$, both fixed by σ, then T_1 should contain an involution τ fixing both A, B, C, and D. But then τ would be an homology, contrary to our assumptions.

There exists a secant of $\mathscr{U}$ which is fixed by both T and α.

Since the number of secants of $\mathscr{U}$ is odd, T fixes a secant l of $\mathscr{U}$. Let $\{A, B\}$ be an orbit of length 2 of T on l and let σ be an involution in T fixing both A and B. By Result 1.4(2), the group $\bar{T}$, induced by T in $F(\sigma)$, either is a triangular group of homologies, or contains two Baer involutions ϕ_1, ϕ_2, together with their product which must be an involutorial homology.

In the former case, the only lines of Π which are fixed by T are the three axes of homologies in $\bar{T}$. So, α fixes each of them.

In the latter, the set of lines of Π which are fixed by T consists of the common pencil of the subplanes $F(\phi_1)$ and $F(\phi_2)$ of $F(\sigma)$ and of the axis l' of $\phi_1\phi_2$. Clearly, α fixes l' and we have the assertion.

Now, (b) yields the final contradiction.

(3) $G \not\simeq Sz(8)$.

Let T be a Sylow 2-subgroup of G. T contains a subgroup T_1 of index two, fixing two points of $\mathscr{U}$. It is well known (see [15, XI, Sect. 3]) that $Z(T)$ is an elementary abelian group of order 8 and that each element of $Z(T)$ is a square in T. Therefore, $Z(T) < T_1$. Let τ be an element of order 4 of T_1. Then $\tau^2 \in Z(T)$ and T_1 induces on $F(\tau^2)$ a collineation group which contains an elementary abelian group of order 8. This contradicts Result 1.4(3(i)).

When G contains involutorial homologies, we have the following result.

THEOREM 3.2. *Let G be a non-abelian simple group and suppose that G contains involutorial homologies. Then one of the following holds*:

(1) $m \equiv 1(4)$ *and either* $G \simeq PSL(2, q)$ *with* q *odd or* $G \simeq PSU(3, q^2)$ *with* $q \equiv 1(4)$ *or* $G \simeq A_7$,

(2) $m \equiv 3(4)$ *and either* G *is a group of homologies with the same centre and the same axis or* $G \simeq PSL(2, q)$, $PSU(3, q^2)$, $Sz(q)$, *or* A_7.

Proof. The proof divides into two parts.

If any two commuting involutorial homologies in G have both distinct centres and distinct axes, then either $G \simeq PSL(2, q)$, where q is odd or $G \simeq PSU(3, q^2)$, where $q \equiv 1(4)$ when $m \equiv 1(4)$, or $G \simeq A_7$.

In this case G has 2-rank $\leqslant 3$ by Theorem A and by Lemmas 1.1 and 2.1(3). When G has 2-rank 3, it must contain both involutorial homologies and Baer involutions by Lemma 1.1(3) and hence it has more than one class of involutions. So, by Result 1.2, $G \simeq PSL(3, q)$, $PSU(3, q^2)$, $PSL(2, q)$, q odd, $PSU(3, 4^2)$, A_7, M_{11}, or M_{12}.

Since G is simple, G must contain distinct commuting involutorial homologies which are conjugate under G. So, G does not leave invariant any point, line, or triangle of Π. By [10, Lemmas 3.3 and 5.3], the centres and the axes of homologies in G generate a subplane Π_0 and G is strongly irreducible on Π_0.

Now, we obtain our result by the following steps.

(1) $G \not\simeq M_{11}$ or M_{12}.

Use the Main Theorem of [19].

(2) $G \not\simeq PSL(3, q)$, q odd.

Suppose that $G \simeq PSL(3, q)$ for q odd. Then Π_0 is a desurguesian plane of order q and all elations of Π_0 are induced by elations of Π (see [17, Theorem C]). Let L be a point of Π_0. We should have $L \in \mathbf{U}$ since L is an elation centre and also, $L \notin \mathbf{U}$ since L is the centre of an involutorial homology. A contradiction!

(3) $G \not\simeq PSU(3, q^2)$ for $q \equiv 3(4)$ and $m \equiv 1(4)$.

Let T be a Sylow 2-subgroup of G. When $m^3 + 1 \equiv 2(4)$, there must be a point P of $\mathscr{U}$ such that $[T: T_P] \leqslant 2$. The group T is wreathed for $q \equiv 3(4)$ (see [9]). Therefore, T contains a Klein group K such that each involution in K is a square in T. So, K is contained in T_P, but this contradicts Lemma 2.2(2).

(4) $G \not\simeq PSU(3, 4^2)$ for $m \equiv 1(4)$.

Use the same argument of (3).

If there exists a Klein group of homologies with the same axis l and the same centre L, then $m \equiv 3(4)$ and either G is a group of (L, l)-homologies or $G \simeq PSL(2, q)$, $Sz(q)$, or $PSU(3, q^2)$ for q even.

We have that $m \equiv 3(4)$ by Lemma 2.2. Let $\Delta = \{l^\alpha : \alpha \in G\}$. If G fixes l, then the group of (L, l)-homologies is normal in G and hence it coincides with G.

Suppose that $|\Delta| > 1$. Let $a = l^{\alpha}$ and $b = l^{\beta}$ with $a \neq b$ and $\alpha, \beta \in G$. Moreover, let $A = L^{\alpha}$. Assume there exists in G an involution σ fixing both a and b. Since $m = 3(4)$, σ is a (C, c)-homology by Lemma 2.1(3). By replacing a with b, if necessary, we may suppose that $c \neq a$. Clearly σ must centralize an involutorial homology τ with axis a. Let D be the centre of τ. By Lemma 1.1, τ is the only involutorial (D, a)-homology and hence $D \neq A$. Since, by our assumptions, there exist distinct involutorial (A, a)-homologies, Lemma 3.1.9 of [7] yields again a contradiction. So, the stabilizer of two distinct lines of Δ must have odd order.

By a result of Bender [1] we then have that $G \simeq PSL(2, q)$, $Sz(q)$, or $PSU(3, q^2)$, where $q = 2^h$, $h \geqslant 2$, and G acts on Δ as $PSL(2, q)$, $Sz(q)$, or $PSU(3, q^2)$ in its usual doubly transitive representation.

Combining Theorems 3.1 and 3.2 we obtain Theorem B.

4. Collineation Groups Which Are Transitive on the Points of an Embedded Unital

Here, we consider the case where G acts transitively on the points of a G-invariant unital $\mathscr{U}$ embedded in Π.

The only known examples are the following (e.g., see [18]):

(1) Π is desarguesian of order q, $\mathscr{U}$ is classical, and $PSU(3, q^2) \leqslant G \leqslant P\Gamma U(3, q^2)$,

(2) Π is desarguesian of order 25, $\mathscr{U}$ is classical, and $G \simeq A_7$,

(3) Π is desarguesian of order 9, $\mathscr{U}$ is classical, and $PSL(2, 7) \leqslant G \leqslant PGL(2, 7)$.

The results of this section seem to strengthen the conjecture that no other example exists.

We use the same notation of the previous sections and *we denote by S the socle of the group G.*

Proposition 4.1. *Assume that G is transitive on the points of $\mathscr{U}$ and that S does not contain involutorial homologies. Then one of the following holds*:

(1) $m \equiv 3(4)$ *and S has odd order*,

(2) $m \equiv 1(4)$ *and either S has odd order or S is non-abelian simple.*

Proof. Let $S = \prod_{i=1}^{n} M_i$, where M_i is a minimal normal subgroup of G.

Case $m \equiv 3(4)$. None of the M_i can contain a simple group by

Lemma 2.1(3). If M_i is an elementary abelian 2-group, then $|M_i| = 2$ by Lemma 2.1(3) again. Let σ be the involution in M_i. The group G should fix the oval Ω_σ, which is impossible. Therefore S has odd order.

Case $m \equiv 1(4)$. Suppose that S has even order and that a component M_i of S has odd order. There exists a component M_j of S which contains an involution σ and σ is centralized by M_i. Therefore M_i leaves invariant the oval Ω_σ.

If $|M_i| = p^h$ for some prime p, then $p \mid |\Omega_\sigma|$ since M_i must be f.p.f. on $\mathbf{U}$ and hence $p \mid m+1$. Since $F(\sigma)$ contains $m(m-1)/2$ external lines of Ω_σ, M_i fixes at least one of these lines, say l. By the transitivity of G on $\mathbf{U}$, if $P \in l \cap \mathbf{U}$, there exists an oval Ω_τ containing P for a suitable involution τ of M_j. M_i centralizes τ and hence M_i fixes Ω_τ. So, M_i fixes P, which is a contradiction.

Therefore, no component of S has odd order. If a component M_i of S is an elementary abelian 2-group, then $|M_i| \leqslant 8$ by Theorem A. Since each point of $\mathcal{U}$ lies in the oval Ω_σ for some $\sigma \in M_i$, we must have $m^3 + 1 \leqslant 7(m+1)$ and this yields $m = 3$, which is impossible.

Now, using Theorem A again, one can easily prove that S must be a simple group.

PROPOSITION 4.2. *Assume that G is transitive on the points of $\mathcal{U}$ and that G contains non-trivial involutorial homologies. Then G is strongly irreducible, S is non-abelian simple, and* $S \leqslant G \leqslant \mathrm{Aut}(S)$.

Proof. By the transitivity of G on $\mathbf{U}$, we have that G does not fix any point, line, or triangle of Π.

If G leaves invariant a subplane Π_0 of Π, then Π_0 and $\mathcal{U}$ cannot have common points and the lines of Π_0 induce a partition on $\mathbf{U}$. So, if Π_0 has order a, then $a^2 + a + 1 = m^2 - m + 1$, that is, $a = m - 1$. This is a contradiction since Π_0 would be a plane of even order admitting involutorial homologies.

Thus, G is strongly irreducible. By [10, Theorem 5.5], $C_G(S) \leqslant S$ and S is either non-abelian simple or elementary abelian of order 9. In the latter case, $|G| \mid 9 \cdot 48$ since $\mathrm{Aut}(S) \leqslant GL(2, 3)$. An easy calculation shows that G cannot act transitively on the points of $\mathcal{U}$.

In the following, *we assume that S is non-abelian simple.*

Several restrictions on the structure of S are given by results of Section 3, but further restrictions can be obtained under the assumption that G is *transitive* on the points of $\mathcal{U}$.

LEMMA 4.3. *If G is transitive on the points of $\mathcal{U}$, then two distinct commuting involutorial homologies in S have both distinct centres and distinct axes.*

Proof. Suppose that S contains a Klein group of homologies with the same axis l and the same centre L. Then $m \equiv 3(4)$ by Lemma 2.2 and S is non-abelian simple by Proposition 4.2.

Let $\Delta = \{l^\alpha : \alpha \in S\}$. Suppose that S fixes l. Then S is a group of (L, l)-homologies and hence G fixes l, which is impossible. Thereore $|\Delta| > 1$. As in the proof of Theorem 3.2, we have that $S \simeq PSL(2, q)$, $Sz(q)$, or $PSU(3, q^2)$, q even, and S acts on Δ in its usual doubly transitive representation of degree $q+1$, q^2+1, and q^3+1, respectively (see [1]).

The involutorial (L, l)-homologies of S, together with the identity, make a group which coincides with the centre of the Sylow 2-subgroup of S fixing l. The centres of two distinct Sylow 2-subgroups of S generate the whole group when $S \simeq PSL(2, q)$ or $Sz(q)$ and the group $SL(2, q)$ when $S \simeq PSU(3, q^2)$ (see [14, 15, 18]).

So, if $a, b \in \Delta$, $a \neq b$, and $P = a \cap b$, then $P \notin \mathbf{U}$ for $S \simeq PSL(2, q)$ or $Sz(q)$ since S cannot fix any point of $\mathscr{U}$.

Let $S \simeq PSU(3, q^2)$ and suppose that $P \in \mathbf{U}$. Then there are $q+1$ lines of Δ through P and no more since S does not fix P. By the transitivity of G on $\mathbf{U}$ there are exactly $q+1$ lines of Δ through each point of $\mathscr{U}$. Counting in two different manners the flags consisting of a point of $\mathscr{U}$ and a line of Δ we obtain the relation $(m^3+1)(q+1) = (q^3+1)(m+1)$ which yields $m = q$. This is impossible since m is odd.

So, as G is transitive on $\mathbf{U}$, the lines of Δ make a partition of $\mathbf{U}$ and hence $|\Delta| = m^2 - m + 1$. This yields the final contradiction since $|\Delta| = q+1$, q^2+1, or q^3+1.

Throughout the rest of this section we shall use the following notation:

t: the cardinality of S_P for $P \in \mathbf{U}$,

s: the number of involutions in S_P,

u: the number of orbits of S on $\mathbf{U}$.

LEMMA 4.4. *If G is transitive on the points of $\mathscr{U}$ and involutions in S are Baer collineations, then we cannot have $S \simeq PSL(2, q)$ for q odd.*

Proof. Suppose that $S \simeq PSL(2, q)$, $q = p^f$, p an odd prime. We can assume that $q \geq 5$ and $m \equiv 1(4)$ by Proposition 4.1.

Let D be a Sylow 2-subgroup of S. Then D is dihedral and $|D| \leq 8$ by Proposition 2.3(2). If $|D| = 8$ and $\tau \in D$ has order 4, then τ induces in $F(\tau^2)$ a Baer collineation with fixed points on Ω_{τ^2}—see the proof of Proposition 2.3. Hence, τ fixes some point of $\mathscr{U}$.

Therefore, as $m^3+1 \equiv 2(4)$, we have that either $u \equiv 2(4)$ and S_P contains a Sylow 2-subgroup of S or u is odd and a Sylow 2-subgroup of S_P is cyclic.

Since $S \leqslant G$ and G is transitive on U, the following relation holds:

$$\frac{q(q-1)(q+1)}{2t} \cdot u = m^3 + 1. \tag{I}$$

Let $\Omega = \{\Omega_\sigma = F(\sigma) \cap \mathrm{U} : \sigma \in S - \{1\},\ \sigma^2 = 1\}$. Since $\Omega_{\sigma_1} \neq \Omega_{\sigma_2}$ for $\sigma_1 \neq \sigma_2$ by Lemma 2.1(4), the cardinality of Ω coincides with the number of involutions in S. By calculating in two different ways the number of flags (L, Ω_σ), where $L \in \mathrm{U}$ and $\Omega_\sigma \in \Omega$, we obtain the following relations:

$$(m^3+1) \cdot s = (m+1) \cdot \frac{q(q+1)}{2}, \qquad q \equiv 1(4), \tag{II}$$

$$(m^3+1) \cdot s = (m+1) \cdot \frac{q(q-1)}{2}, \qquad q \equiv 3(4). \tag{II$_1$}$$

Case $q \equiv 1(4)$. By [14, II, Hauptzatz 8.27], we have the following possibilities for S_P:

(a) $S_P \leqslant K$, where K is cyclic of order $(q-1)/2$,

(b) $S_P \leqslant N_S(K)$, where K is cyclic of order $(q \pm 1)/2$, $S_P \nleqslant K$,

(c) $S_P \leqslant N_S(P)$, where P is a Sylow p-subgroup of S,

(d) $S_P \simeq PSL(2, p^m)$ or $PGL(2, p^m)$, where $m \mid f$ or $2m \mid f$, respectively,

(e) $S_P \simeq A_4$, S_4, or A_5.

Using (I), we obtain from (II)

$$m+1 = \frac{(q-1) \cdot u \cdot s}{t}. \tag{III}$$

Put $k = (u \cdot s)/t$. Now, (II) yields

$$k = \frac{3 \pm \sqrt{(2q(q+1)/s) - 3}}{2(q-1)}. \tag{IV}$$

A straightforward calculation shows that $\sqrt{(2q(q+1)/s) - 3} \geqslant 3$ in all cases (a)–(e). So, we can assume that (IV) holds with the sign "+". The case $q = 5$ cannot occur by (II). Thus, by (IV) we have that

$$k < 1/2 \qquad \text{for} \quad s \geqslant 3. \tag{$*$}$$

The cases (b) and (e) cannot occur. Use $(*)$ and the fact that $u \equiv 2(4)$ when S_P contains a Klein group.

Now, we consider the cases (a), (c), and (d) for f odd. Let σ be an involution in S. The oval Ω_σ is left invariant by $C_S(\sigma)=N_S(C)$, where C is cyclic of order $(q-1)/2$ and contains σ. If $P\in\Omega_\sigma$, then $\sigma\in S_P$. From the structure of S we infer that $|S_P\cap C|=(|S_P|, (q-1)/2)$ in cases under consideration. So, $|S_P\cap C|$ is independent from the choice of $P\in\Omega_\sigma$. Since C is cyclic, $S_P\cap C$ is a fixed subgroup of C for all $P\in\Omega_\sigma$. Thus, $S_P\cap C$ fixes Ω_σ pointwise and hence $|S_P\cap C|=2$ by Lemma 2.1(4).

In case (a), $S_P\cap C=S_P$ and hence $t=2$ and $s=1$. We then have from (IV)

$$u=\frac{3+\sqrt{2q(q+1)-3}}{q-1}$$

but this equation does not have any integer solution.

In case (c), $t=2\cdot p^h$ and $s=p^h$ for some $h\geqslant 1$. This yields $k\geqslant 1/2$, in contrast with $(*)$.

In case (d), we must have $(p^m-1)/2=2$, that is, $S_P\simeq PSL(2,5)\simeq A_5$. This occurrence has been already excluded.

The case (d) for f even cannot occur. As before, let σ be an involution in S and let $N=N_S(C)=C_S(\sigma)$.

Here, $8\mid q-1$. Put $\bar{N}=N/\langle\sigma\rangle$. $\bar{N}$ is a dihedral group of order $(q-1)/2$ with cyclic stem $\bar{C}$ and acts on $F(\sigma)$ as a collineation group leaving Ω invariant. As we have seen in the proof of Proposition 2.3(2) the involution of $\bar{C}$ acts as a Baer collineation in $F(\sigma)$. By Result 1.4(2), one among the two conjugacy classes of involutions in $\bar{N}-\bar{C}$, say $\bar{H}$, consists of homologies. Let $\bar{D}$ be the subgroup of (odd) order $(q-1)/8$ of $\bar{C}$. Since $\bar{D}=\langle\sigma_1,\sigma_2\rangle$ for suitable elements $\sigma_1,\sigma_2\in\bar{H}$ and $\bar{H}\subset\langle\sigma_1,\bar{D}\rangle$, all the axes of homologies in $\bar{H}$ pass through the same point M of $F(\sigma)$. Furthermore, either all these axes are external to Ω_σ or all these axes are secant to Ω_σ. So, $(m+1)/2\geqslant(q-1)/8$ in any case, that is, $m+1\geqslant(q-1)/4$. This inequality, together with (II), yields

$$s\leqslant\frac{8q^2+8q}{q^2-14\cdot q+61}. \tag{V}$$

When $S\simeq PSL(2,p^m)$ or $PGL(2,p^m)$ for f even, we may assume that $s>15$ since the cases $S_P\simeq A_4$, S_4, or A_5 have been already excluded. So, (V) is satisfied only for $q=25$ and $S_P\simeq PGL(2,5)$. In this case $s=25$ and (II) is unsatisfied.

The case $q\equiv 3(4)$ can be excluded in a similar manner.

LEMMA 4.5. *If G is transitive on the points of $\mathscr{U}$ and involutions in S are Baer collineations, then we cannot have $S\simeq PSL(2,8)$.*

Proof. By the same counting argument which yieds relation (II) of

Lemma 4.4 one can prove that $m=3$ or 5. The case $m=3$ cannot occur by Lemma 2.1(3).

Assume that $m=5$. A Sylow 2-subgroup T of S fixes a secant l of $\mathcal{U}$. Since $|l \cap \mathrm{U}|=6$ and T is elementary abelian of order 8, T must contain a non-trivial element fixing at least four points on $l \cap \mathrm{U}$. This is a contradiction since an involution in T fixes at most two points on $l \cap \mathrm{U}$.

LEMMA 4.6. *If G is transitive on the points of $\mathcal{U}$, S contains involutarial homologies and $S \simeq PSL(2, q)$ for q odd, then Π is desarguesian of order* 9, *$\mathcal{U}$ is classical, and $S \simeq PSL(2, 7)$.*

Proof. Suppose there are two distinct involutorial homologies $\sigma_1, \sigma_2 \in S$ with the same axis l. Then $H=\langle C_S(\sigma_1), C_S(\sigma_2)\rangle$ fixes l. Now, use [14, II, Hauptsatz 8.27].

When $q \equiv 3(4)$ we have that either $H=S$ or $S \simeq PSL(2,7)$ and $H \simeq S_4$. In the former case S is a group of (L, l)-homologies in contrast with Lemma 4.3. In the latter, $\langle \sigma_1^\tau : \tau \in H\rangle$ contains a Klein group of homologies with axis l. This contradicts Lemma 4.3 again.

When $q \equiv 1(4)$, either $H=S$ or $S \simeq PSL(2,5)$ or $PSL(2,9)$. Again, it is easily seen that none of these cases can occur. So, distinct involutorial homologies have distinct axes.

From now forward the proof partially overlaps with that of Lemma 4.4. As in Lemma 4.4, relations (I), (II), and $(\mathrm{II})_1$ hold. Here, we consider flags consisting of a point of $\mathcal{U}$ and of the axis of an involutorial homology in S. Furthermore, the Sylow 2-subgroups of S_P are cyclic by Lemma 2.2(2).

Case $q \equiv 1(4)$. Here, only the cases (a), (b), and (c) of Lemma 4.4 are admissible for S_P. Clearly, relations (III), (IV), and $(*)$ of Lema 4.4 still hold. Using $(*)$ we may exclude the case (b).

Consider the case (c). We have that $S_P=Q \cdot D$ is a Frobenius group. Q is the Frobenius kernel and has order p^h with $h \geqslant 1$, while D is a Frobenius complement of order d dividing $(q-1)/2$.

Now, (II) yields

$$(m^2-m)\,p^h+(p^h-1)-q(q-1)/2=0$$

and hence $d \mid m^2-m$. On the other hand, (III) yields $m+1=(q-1)\cdot u/d$ since $s=p^h$. So, $((q-1)/d) \mid m+1$. We then have

$$\left(\frac{q-1}{2d}, d\right)=1 \qquad \text{for} \quad m \equiv 1(4), \tag{1.a}$$

$$\left(\frac{q-1}{d}, \frac{d}{2}\right)=1 \qquad \text{for} \quad m \equiv 3(4). \tag{1.b}$$

Using (III), we obtain from (II)

$$(q-1)^2u^2 - 3(q-1)t\cdot u - \frac{q(q-1)}{2}\cdot t^2 - (q-3)t^2 = 0$$

So, $((q-1)/2) \mid t^2(q-3)$ or also, $((q-1)/2) \mid d^2p^{2h}(q-3)$ and hence

$$q-1 \mid 4d^2. \tag{2}$$

Now, (2) yields $((q-1)/2d) \mid 2d$ and hence $((q-1)/d) \mid 2$ for $m \equiv 1(4)$ by (1.a)—notice that $(q-1)/2d$ is odd. Likewise, $((q-1)/d) \mid 8$ for $m \equiv 3(4)$ by (1.b). So, $d = (q-1)/2^i$, where $i = 1, 2$, or 3. As $((q-1)/2^i) \mid p^h - 1$, one can see that we must have $p^h = q$ except when $q = 9$, $h = 1$, $d = 2$ or $q = 49$, $h = 1$, $d = 6$. Exceptions may be excluded using (II).

Therefore, (IV) yields

$$u = \frac{3 \pm \sqrt{2q-1}}{2(q-1)}\cdot\frac{t}{s} = \frac{3 \pm \sqrt{2q-1}}{2^{i+1}}, \tag{IV$'$}$$

where $i = 1, 2$, or 3. Since $u \geqslant 1$, (IV$'$) holds with the sign "+". We have that $G \leqslant P\Gamma L(2, q)$ by Proposition 4.2. Since G is transitive on U, $u \leqslant [P\Gamma L(2, q){:}\ PSL(2, q)] = 2\cdot\log_p q$ and hence

$$q^{2^{i+2}} \geqslant p^{3+\sqrt{2q-1}}. \tag{VI}$$

When $i = 3$, (VI) is satisfied only for

$$\begin{array}{ll} p \leqslant 509, & q = p \\ p \leqslant 43, & q = p^2 \\ p \leqslant 13, & q = p^3 \\ p \leqslant 7, & q = p^4 \\ p = 5, & p^5 \leqslant q \leqslant p^6 \\ p = 3, & p^7 \leqslant q \leqslant p^9. \end{array}$$

Since u must be an integer a straightforward calculation shows that the only admissible value is $p = q = 421$. This value is excluded for $d = (q-1)/8$ is not an integer in this case.

When $i = 2$, in the same manner one can see that the only admissible value is $p = q = 13$. In this case $u = 1$, $m = 3$ by (III) and $S \simeq PSL(2, 13)$. The group $PSL(2, 13)$ cannot act in a projective plane of order 9 (e.g., see [12, Theorem C]).

Lastly, when $i = 1$, we must have $p = q = 13$, $u = 2$, $m = 3$. This case is excluded as above.

In case (a), the same argument of case (c) works using t instead of d.

Case $q \equiv 3(4)$. Here, the analogous of $(*)$ holds only for $q > 7$. All possibilities for $q > 7$ may be excluded as in the case $q \equiv 1(4)$.

If $q = 7$, $(\mathrm{II})_1$ implies that either $m = 5$, $s = 1$ or $m = s = 3$. The argument for case (a) works also for $q = 7$. So, the former case cannot occur. For what concerns the latter case, $\mathscr{U}$ is either a *Ree* or a *classical* unital since $|PSL(2, 7)| \mid |\mathrm{Aut}(\mathscr{U})|$ (see [3, Sect. C]). The automorphism group of a Ree unital on 28 points is isomorphic to $\mathrm{Ree}(3) \simeq P\Gamma L(2, 8)$ (e.g., see [15]) and hence it does not contain $PSL(2, 7)$. So, $\mathscr{U}$ is a classical unital and Π is desarguesian since $\mathscr{U}$ is uniquely embeddable—again see [3, Sect. B].

LEMMA 4.7. *If G is transitive on the points of $\mathscr{U}$, S contains involutorial homologies and $S \simeq PSU(3, q^2)$, q odd, then Π is a desarguesian plane of order q^2 and $\mathscr{U}$ is a classical unital.*

Proof. By Proposition 4.2, G is strongly irreducible.

Suppose that $q \neq 3$. Then by [11, Proposition 5.3], we may assume—up to duality—the existence of a G-invariant unital $\mathscr{W}$ embedded in Π on which S acts in the classical way. Let $\mathbf{W}$ be the point-set of $\mathscr{W}$ and suppose that $\mathbf{U} \cap \mathbf{W} = \varnothing$. Let $P \in \mathbf{W}$ and let l be a tangent of $\mathscr{U}$ through P. Put $L = l \cap \mathbf{U}$. If l is a secant of $\mathscr{W}$, then S_l contains a triangular group of homologies. This contradicts Lemma 2.2(2), since S_l fixes L. So, l meets $\mathbf{W}$ only in the point P. Since G is transitive on $\mathbf{U}$, there exists an involutorial homology $\sigma \in S$ whose axis p passes through L. We have that σ fixes l and $\mathbf{W}$ and hence σ fixes the point P. So, P is the centre of σ. Indeed $P \notin p$ since $p \neq l$. This is impossible because the centre of σ cannot lie in $\mathbf{W}$. Therefore $\mathbf{U} \cap \mathbf{W} \neq \varnothing$ and hence $\mathscr{U} = \mathscr{W}$.

Suppose that $q = 3$. Using arguments similar to those of Lemma 4.6 one can prove that two distinct involutorial homologies have distinct axes. Now, counting in two different ways the flags consisting of a point $P \in \mathbf{U}$ and of the axis of an involutorial homology in S, we obtain the relation $(m^2 - m + 1) \cdot s = 63$, where 63 is the number of involutions in S. So, either $m = 3$ or $m = 5$. The case $m = 5$ cannot occur by Theorem 3.2(1). Therefore, $m = 3$ and $\mathscr{U}$ is classical by [13].

Proof of Theorem C. Let S be the socle of G. Combining Proposition 4.1, Theorem B, and Lemmas 4.4 and 4.5 we have that either S contains involutorial homologies or $S \simeq A_7$.

In the former case, Proposition 4.2, Theorem 3.2, and Lemmas 4.3, 4.6, and 4.7 show that either (*j*) or (*jj*) holds or $S \simeq A_7$. So, we have only to investigate the case $S \simeq A_7$.

The usual argument on flags yields the relation $(m^2 - m + 1) \cdot s = 105$, where 105 is the number of involutions in S (e.g., see [5, Appendix 3]).

Therefore, either $m=3$ or $m=5$. Since G is transitive on $\mathbf{U}$, we have the following possible cases:

(a) $m=3$, $s=15$, $t/u=90$,

(b) $m=5$, $s=5$, $t/u=20$.

Using Appendeix 3 of [5], which explains the structure of the subgroups of A_7, it is not hard to see that only the case (b) can occur for S_P of type $5 \cdot 4$ and $u=1$.

It remains to show that $C_G(S) = \langle 1 \rangle$ when $S \simeq A_7$ and involutions in S are Baer collineations. If $Q, R \in \mathbf{U}$ and $Q \neq R$, then S_Q and S_R have at most one involution in common—again, see [5, Appendix 3]. So, if $\sigma_1, ..., \sigma_5$ are the five involutions of S_P, $P \in \mathbf{U}$, then the ovals $\Omega_{\sigma_1}, ..., \Omega_{\sigma_5}$ have exactly the point P in common. If $\tau \in C_G(S)$ then τ leaves invariant each of the ovals Ω_{σ_i}, $1 \leqslant i \leqslant 5$, and hence τ fixes P. Therefore, τ fixes $\mathscr{U}$ pointwise and hence $\tau = 1$.

Theorem D is an easy corollary of Theorem C.

Proof of Theorem D. If Π has order 25 and $G \simeq A_7$, then G_P is a Frobenius group of order 20, which is not maximal in G (e.g., see [5, Appendix 3]). It could be $G \simeq S_7$, but it is an easy exercise to show that G_P is not maximal in G also in this case.

If Π has order 9 and $G \simeq PSL(2, 7)$, then G_P is dihedral of order 6. Such a group is not maximal in G since it is contained in a subgroup of G isomorphic to S_4. When $G \simeq PGL(2, 7)$, G_P is dihedral of order 12 and is maximal in G.

Now, Theorem C gives our assertion.

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